V. Conclusions

The numerical performance of a new LMI-based design technique for robust $\mathcal{H}_2$ controllers is analyzed for systems with real parametric uncertainty. This technique iterates between solving 1) LMIs for the analysis parameters with the controller values fixed and 2) LMIs for the controller values with the multiplier parameters fixed. The approach is, thus, quite similar to the D–K iteration typically used for μ synthesis. The results from this new algorithm were validated using a previously published benchmark example. Note that, from previous experience with both gradient optimization and LMI synthesis, an important potential advantage of the LMI approach is the low implementation overhead associated with this optimization, especially given the simplicity of current semidefinite programming interfaces.[14] This advantage also greatly simplifies the extension of the algorithm to include more general stability multipliers, as discussed in Ref. 16. Because controller synthesis is a nonconvex optimization problem, this iterative algorithm is generally not guaranteed to converge globally. However, for the numerical results presented on two standard benchmark problems, the synthesis algorithm is shown to rapidly converge close to the local optimal solution (typically in 3–4 iterations).

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New Class of Intermediate-Thrust Arcs for Trajectories in Newtonian Gravitational Fields

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Introduction

As is known, the optimal trajectories of a rocket moving with constant exhaust velocity and limited mass-flow rate in a Newtonian gravitational field may consist of arcs of null thrust (NT), intermediate thrust (IT), and maximum thrust (MT).[1] The IT arcs represent singular arcs, and their appearance is a degenerate case with added difficulties in determining the controls.[2] All analytical solutions for the IT arcs can be divided into two classes: solutions for cases of free-and fixed-time problems. In the first case, Lawden's spiral,[1,2] spiral-shaped trajectories[3] and two classes of alternative spirals[4] in which the expression for the radius vector is the same as for Lawden's spiral are known. For the fixed-time problems some spiral,[4,5] circular,[6,7] and spherical trajectories[6,8] were found. It was shown that Lawden's spiral is nonoptimal.[4,5,9,10] Other solutions, taking into account the minimizing functional, do not satisfy the transversality condition[3] or Robbins's condition.[4,6–9] However, some spirals can satisfy these conditions and may be used for solving the interorbital transfer problem.[4] Although all of these results represent some progress in solving the problem, questions about existence of other solutions to IT arcs, their optimality, applicability, etc. have not been thoroughly studied. The analysis of known works shows that to verify optimality and applicability of IT arcs for the specific problem, it is necessary to investigate them for conditions of existence, optimality, transversality, and continuity at the switching points. The present work is devoted to developing this point of view on the basis of new spiral trajectories for the IT arcs.

Statement of the Problem

The equations of motion in the Newtonian field may be given in the form[1]

$$\dot{\boldsymbol{v}} = (cm/M)\boldsymbol{u} - (\mu/r^3)\boldsymbol{r}, \qquad \dot{\boldsymbol{r}} = \boldsymbol{v}, \qquad \dot{M} = -m$$

where $\boldsymbol{r} = (r, 0, 0)$ is the radius vector of the spacecraft with origin at the attracting center, $\boldsymbol{v} = (v_1, v_2, v_3)$ is the velocity vector, $\boldsymbol{u} = (u_1, u_2, u_3)$ is a unit thrust vector, M is the mass of the spacecraft, m $(0 \leq m \leq \bar{m})$ is the mass-flow rate, and c is the exhaust velocity. Here the components of all vectors are given in a spherical coordinate system (r, θ, and δ) with the origin at the attracting center.[7] The known theory of optimum trajectories does not allow use of constraints in the form of inequalities and, therefore, we transform the inequality for the m to an equality via introduction of a slack variable.[1] For this it is required that m satisfies the equality $m(\bar{m} - m) - \alpha^2 = 0$. Additionally, for the components of the thrust vector the following equality will hold: $u_1^2 + u_2^2 + u_3^2 = 1$. The m, α, and u_i $(i = 1, 2, 3)$ are piecewise continuous control functions. For simplification, all variables are denoted by x_i $(i = 1, \ldots, 7)$, that is, the components of $\boldsymbol{v}$ are denoted by x_1, x_2, and x_3; the components of $\boldsymbol{r}$ are denoted by x_4, x_5, and x_6; and the rocket mass is denoted by x_7. It is assumed that the initial conditions $x_j = x_{j0}$ $(j = 1, \ldots, 7)$ and final conditions $x_n = x_{n1}$, $n = 1, \ldots, l$, and $l < 7$ are given. It is required to find the time histories of m, α, and u_i such that x_i would satisfy the equations of motion, the preceding control constraints, the initial and final conditions, and minimize the given

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functional[1] $J(x_{l+1,1}, x_{l+2,1}, \ldots, x_{7,1})$. The costate equations are given by[1]

$$\dot{\boldsymbol{\lambda}} = -\boldsymbol{\lambda}_r, \qquad \dot{\boldsymbol{\lambda}}_r = (\mu/r^3)\boldsymbol{\lambda} - 3(\mu/r^5)(\boldsymbol{\lambda r})\boldsymbol{r}$$

$$\dot{\lambda}_7 = (cm/M^2)\boldsymbol{u}\lambda$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ is the primer vector,[1]

$$\boldsymbol{\lambda}_r = [\lambda_4, \lambda_1(v_2/r) + (\lambda_2/r)(v_3 tg\delta - v_1) - \lambda_3(v_2/r)tg\delta + (\lambda_5/(r\cos\delta)), \lambda_1(v_3/r) - \lambda_3(v_1/r) + (\lambda_5/r)]$$

is the vector conjugate (adjoint) to $\boldsymbol{r}$; λ_4, λ_5, and λ_6 are multipliers conjugate to r, θ, and δ, correspondingly; and λ_7 is a multiplier conjugate to M. It can be shown by the Weierstrass conditions[1,7] or the Pontryagin maximum principle[2,11] that $\boldsymbol{u} = \boldsymbol{\lambda}/\lambda$ and the state and costate equations may be rewritten as[7]

$$\dot{\boldsymbol{v}} = (cm/M)(\boldsymbol{\lambda}/\lambda) - (\mu/r^3)\boldsymbol{r}, \qquad \dot{\boldsymbol{r}} = \boldsymbol{v}, \qquad \dot{M} = -m$$

$$\dot{\boldsymbol{\lambda}} = -\boldsymbol{\lambda}_r, \qquad \dot{\boldsymbol{\lambda}}_r = (\mu/r^3)\boldsymbol{\lambda} - 3(\mu/r^5)(\boldsymbol{\lambda r})\boldsymbol{r}$$

$$\dot{\lambda}_7 = (cm/M^2)\lambda \tag{1}$$

which is of a canonical type

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i}, \qquad \dot{\lambda}_i = -\frac{\partial H}{\partial x_i}, \qquad (i = 1, \ldots, 7)$$

with the Hamiltonian

$$H = -(\mu/r^3)\boldsymbol{\lambda r} + \boldsymbol{\lambda}_r \boldsymbol{v} + \chi m$$

where $\chi = (c/M)\lambda - \lambda_7$ is the switching function. Note[1] that if $\alpha = 0$ then $m = 0$ or $m = \bar{m}$, and if $\alpha \neq 0$ the intermediate values of m are accessible $0 < m < \bar{m}$. As is well known from the Weierstrass conditions[1,7] or the Pontryagin maximum principle,[2,11] $\chi \leq 0$ on the NT arcs where $m = 0$, $\chi = 0$ on the IT arcs where $0 < m < \bar{m}$, and $\chi \geq 0$ on the MT arcs where $m = \bar{m}$. The IT arcs will now be considered.

Some Relations for the IT Arcs

For the IT arcs in the polar case the following first integrals of Eqs. (1) are known[6,11]:

$$\lambda_1(v_2^2/r - \mu/r^2) - \lambda_2(v_1 v_2/r) + \lambda_4 v_1 + \lambda_5(v_2/r) = C \tag{2}$$

$$\lambda_1 v_1 + \lambda_2 v_2 - 2\lambda_4 r + c\lambda \ln(M_0/M) - 3Ct = C_1 \tag{3}$$

$$\lambda_7 M = C_2 \tag{4}$$

$$\lambda_5 = C_3 \tag{5}$$

where C, C_1, C_2, and C_3 are integration constants. From the equalities $\dot{\chi} = \chi^{(2)} = \chi^{(3)} = 0$ we find the following invariant relations[12]:

$$\lambda_1\lambda_4 + \lambda_1\lambda_2(v_2/r) - \lambda_2^2(v_1/r) + \lambda_5(\lambda_2/r) = 0 \tag{6}$$

$$\lambda_4^2 + [\lambda_1(v_2/r) - \lambda_2(v_1/r) + \lambda_5/r]^2 = \lambda^2(\mu/r^3) - 3\lambda_1^2(\mu/r^3) \tag{7}$$

$$(\lambda^2 - 5\lambda_1^2)v_1 + 2\lambda_1(\lambda_1 v_1 + \lambda_2 v_2) - 4\lambda_1\lambda_4 r = 0 \tag{8}$$

Excluding λ_4, λ_5, and $(\lambda_1 v_2 - \lambda_2 v_1)$ from Eqs. (2) and (5–7), we obtain

$$C^2\lambda^4 r^4 + 6\mu C\lambda^2\lambda_1^3 r^2 + \mu C_3^2\lambda^2\left[3(\lambda_1^2/\lambda^2) - 1\right]r + 9\mu^2\lambda_1^6 = 0 \tag{9}$$

where $\lambda =$ const, $\lambda_1 = \lambda\sin\varphi$, $\lambda_2 = \lambda\cos\varphi$, and φ is the angle between $\boldsymbol{\lambda}$ and perpendicular to $\boldsymbol{r}$. Some solutions of Eq. (9) were presented in Ref. 4. We will now obtain new solutions of this equation for IT arcs and study the necessary conditions of their optimality and applicability.

Spiral IT Arcs

In this section we will obtain the analytical solutions of the canonical system (1) in the case when the minimizing functional is explicitly independent of flight angular distance. The transversality condition for the final time yields[1]

$$\lambda_{51} = -\frac{\partial J}{\partial \theta_1} = 0 \tag{10}$$

where θ_1 is the final value of the polar angle. Then taking into account the cyclic integral (5) we can conclude that $\lambda_5 = 0$ over the entire length of the trajectory. Consequently, from Eq. (9) we obtain

$$r = [-(3\mu\lambda/C)\sin^3\varphi]^{\frac{1}{2}} \tag{11}$$

which is valid for $C < 0$, $\sin\varphi > 0$ or $C > 0$, $\sin\varphi < 0$. Excluding the multiplier λ_4 from relations (6–8) we find the components of velocity

$$v_1 = 2\frac{\lambda_2}{(3\lambda^2 - 5\lambda_1^2)}q_1 \tag{12}$$

$$v_2 = \frac{5\lambda^2 - 7\lambda_1^2}{\lambda_1(3\lambda^2 - 5\lambda_1^2)}q_1 \tag{13}$$

where

$$q_1 = \left[Cr\lambda_1 + \lambda_1^2(\mu/r)\right]^{\frac{1}{2}}$$

and consequently the polar angle and time in terms of the angle φ

$$\theta = -\tfrac{15}{4}\cot\varphi - \tfrac{21}{4}\varphi + \theta_0 \tag{14}$$

$$t = \frac{9}{4}\int \frac{(5\sin^2\varphi - 3)\,\mathrm{d}\varphi}{[r\lambda^2\mu(1 - 3\sin^2\varphi)]^{\frac{1}{2}}} + t_0 \tag{15}$$

Substituting λ_4 from Eq. (8), v_1 from Eq. (12), and v_2 from Eq. (13) into integrals (3) and (4) allows us to find the mass M and conjugate multiplier λ_7 in the form

$$M = M_0 e^{h(\varphi)} \tag{16}$$

$$\lambda_7 = (C_2/M_0)e^{-h(\varphi)} \tag{17}$$

where

$$h(\varphi) = \frac{1}{c\lambda}\left[\lambda_2 q_1 \frac{(5\lambda_1^2 - \lambda^2)}{\lambda_1(3\lambda^2 - 5\lambda_1^2)} - 3Ct - C_1\right]$$

The mass-flow rate can be found from the condition $\chi^{(4)} = 0$ in the form[4,12]

$$m = M\lambda\frac{\lambda_1(\lambda^2 - 5\lambda_1^2)(v^2 - \mu/r)}{c\lambda_1^2 r(3\lambda^2 - 5\lambda_1^2)} + \frac{q_2(q_2 - \lambda_1 v_1 + \lambda_2 v_2) + \lambda_1 q_3}{c\lambda_1^2 r(3\lambda^2 - 5\lambda_1^2)} \tag{18}$$

where

$$q_2 = 3\lambda_1\lambda_2 v_2 + (\lambda_1^2 - 2\lambda_2^2)v_1 + 2\lambda_2\lambda_5$$

$$q_3 = \left[2\lambda^2 v_1 - 5\lambda_1(\lambda_1 v_1 + \lambda_2 v_2)\right]v_1$$

The system (11–18) presents parametric equations of motion on spiral IT arcs. For $\sin\varphi < 0$, $C > 0$, the solutions obtained will satisfy Robbins's condition[12] and can be used in problems on minimization of the characteristic velocity. When φ is changed from $\varphi = 0$ to $\varphi_1 = \arcsin[-(1/\sqrt{3})]$, the value r is increased from $r = 0$ to $r = \frac{1}{3}\sqrt{(\lambda\mu/C)}$, $(C > 0)$, whereas the polar angle θ is decreased from $\theta = +\infty$ to $\theta = \theta_0 + \frac{15}{8} - \frac{21}{4}\varphi_1$. It is clear that $-(1/\sqrt{3}) < \sin\varphi < 0$ and that the direction of motion depends on the sign of q_1. Therefore, taking into account that

$$\frac{\mathrm{d}\theta}{\mathrm{d}\varphi} = \frac{15}{4\sin^2\varphi} - \frac{21}{4} > 0$$

and

$$\frac{dr}{d\varphi} = -\frac{9\mu\lambda}{C}\frac{\sin^2\varphi\cos\varphi}{2r} < 0$$

it may be shown that for $q_1 > 0$ ($d\theta/dt < 0$) the motion on an outward spiral from the pole in the clockwise direction and departing to distance $r = \frac{1}{3}\sqrt{(\lambda\mu/C)}$, at which φ and θ are decreased, takes place. For $q_1 < 0$, ($d\theta/dt > 0$) the motion on an inward spiral to the pole in the counterclockwise direction, at which θ and φ are increased, takes place. If φ is changed to $\varphi + \pi$ (or $\varphi - \pi$), then $dr/d\varphi > 0$ and so, for $q_1 > 0$ ($d\theta/dt > 0$), the r, θ, and φ are increased, and motion on an outward spiral from the pole in the clockwise direction takes place. For $q_1 < 0$ ($d\theta/dt < 0$) values of r, θ, and φ are decreased, and the motion on an inward spiral to the pole in the counterclockwise direction takes place. Note that analysis of the solutions for components of velocity shows that the angle between the direction of motion and the perpendicular to $\boldsymbol{r}$ is $\sim 0.4\varphi$.

Example

As an example, on the basis of results just obtained, consider the problem of determination of the optimal fixed-time transfer trajectory between coaxial elliptical orbits with eccentricities e_1 and e_2 and semilatus recta p_1 and p_2 by intermediate thrust force. Longitudes of pericenters ω_1 and ω_2 are assumed to be zero. The characteristic velocity is considered the minimizing functional. In this case $\lambda_7(t_1) = -(\partial J/\partial M_1) = c/M_1$, where t_1 is the final time of the transfer. For convenience, it is assumed that the transfer trajectory consists of an NT, IT, and NT sequence of arcs. Then, it must contain two switching points. For determining this trajectory consider the conditions of continuity of the radius and velocity vectors at the points

$$r_1^2 = -3(\mu/C)s_i^3 = p_i^2/(1 + e_i\cos f_i)^2 \tag{19}$$

$$2\frac{\cos\varphi_i z_i}{(3 - 5s_i^2)} = e_i\mu^{\frac{1}{2}}\frac{\sin f_i}{p_i^{\frac{1}{2}}} \tag{20}$$

$$\frac{z_i(5 - 7s_i^2)}{s_i(3 - 5s_i^2)} = \frac{\mu^{\frac{1}{2}}}{p_i}(1 + e_i\cos f_i) \tag{21}$$

where

$$s_i = \sin\varphi_i, \qquad z_i = \left[Cr_i s_i + s_i^2(\mu/r_i)\right]^{\frac{1}{2}} \qquad i = 1, 2$$

where f_i is the true anomaly. The index i represents the values of corresponding variables at the first and second switching points. First, s_i^2 should be found from

$$\left(\tfrac{3}{2} - b_i^2/8\right)\left\{-b_i^2(1 + e_i^2) + \left[b_i^2(e_i/2)\right]^2 - 2b_i\left[4(e_i^2 + 1) - b_i^2e_i^2\right]^{\frac{1}{2}} + 4\right\}^{\frac{1}{2}} = a_i^2(1 - 3s_i^2) \tag{22}$$

where

$$a_i = \frac{5 - 7s_i^2}{5 - 3s_i^2}, \qquad b_i = \frac{5 - 7s_i^2}{s_i\cos\varphi_i}$$

Then unknowns f_i and C may be found from

$$\sin f_i = \frac{\left\{-b_i + \left[4(e_i^2 + 1) - b_i^2e_i^2\right]^{\frac{1}{2}}\right\}}{2b_i(1 - b_i^2/4)} \tag{23}$$

$$C = -3\mu\left[s_1^3(1 + e_1\cos f_1)^2\right]/p_1 \tag{24}$$

It is clear that the values of φ_i and f_i depend only on e_i and are independent of the semilatus recta of boundary orbits; however, the following equality must be satisfied:

$$\frac{s_1^3}{s_2^3}\frac{(1 + e_1\cos f_1)^2}{(1 + e_2\cos f_2)^2} = \frac{p_1^2}{p_2^2} \tag{25}$$

Because $\omega_1 = \omega_2 = 0$, to find θ_0 it is necessary to equate the value of $f_1(e_1, \varphi_1)$ and θ_1. Then $\theta_0 = f_1 + \frac{15}{4}\cot\varphi_1 + \frac{21}{4}\varphi_1$. For determining the primer vector for the boundary orbits, it is sufficient to find its coefficients B_i and D_i from the equations evaluated at the switching points[1]:

$$\sin\varphi_i = B_i e_i\sin f_i + CI_{2i}(f_i, e_i) \tag{26}$$

$$\cos\varphi_i = B_i(1 + e_i\cos f_i) + \frac{D_i C I_{2i}(f_i, e_i)}{1 + e_i\cos f_i} \tag{27}$$

where

$$I_{2i} = \frac{\cot f_i}{e_i(1 + e_i\cos f_i)} + \frac{1 + e_i\cos f_i}{e_i\sin f_i}I_{1i}$$

$$I_{1i} = \sin f_i\int\frac{df}{\sin^2 f(1 + e_i\cos f)^2}$$

For the application of results obtained in this section, the transfer between elliptical orbits with eccentricities $0.21 \leq e_i \leq 0.27$ ($I = 1, 2$) has been considered. In particular, for transfer between elliptical orbits with $e_1 = 0.21$, $p_1 = 9000.0$ km, $e_2 = 0.27$, and $p_2 = 9731.701$ km, the numerical solutions of Eqs. (19–22) and the values of other variables, including the results for impulsive transfer, are the following: $r_1 = 7628.3460$ km, $r_2 = 7818.422$ km, $\sin f_1 = 0.51658$, $\sin f_2 = 0.42253$, $\sin\varphi_1 = -0.48721$, $\sin\varphi_2 = -0.49527$, $t = 105.5543$ s, $C = 0.002376547$, $W_{IT} = 0.694582$, $e_{tr} = 0.28$, $p_{tr} = 9548.50$ km, and $\Delta V_{imp} = 0.28593$. Here W_{IT} is the ratio of characteristic velocity for the transfer by IT arc to the local circular velocity, and e_{tr} and p_{tr} are, respectively, the eccentricity and the semilatus rectum of the Keplerian ellipse for the impulsive transfer.[13]

Numerical results show that the system (11–18) does not have solutions for $0.01 < e_i < 0.21$ and $0.28 < e_i < 0.41$. For $e_i > 0.41$, we have $\sin\varphi = -0.7486$. However, because $-(1/\sqrt{3}) < \sin\varphi < 0$, only values of $0.21 < e_i < 0.28$ can be used for a given problem. Note that values of e_i have been determined independently of semilatus recta of the boundary orbits. The ratio between characteristic velocities for transfers via IT and impulsive thrust satisfies the inequality $1.6 < W_{IT}/\Delta V_{imp} < 5$. Hereafter we will show how to use Eqs. (19–27) to determine the transfer trajectory.

1) It is necessary to find the angles φ_i and f_i for any e_i ($i = 1, 2$), respectively [from Eqs. (22) and (23)].

2) At known e_1, φ_1, and f_1, the C and, consequently, r can be found depending on p_1 [from Eqs. (19) and (22)].

3) For e_i of the specific orbits using corresponding values of φ_i and f_i (where $i = 1, 2$), and also p_1 and C, found corresponding to point b, the r_2, p_2, t, and W_{IT} may be determined [from Eqs. (15) and (18–25)].

4) Then the coefficients B_i and D_i can be easily found from Eqs. (26) and (27).

5) Find the values of the characteristic velocity of the impulsive transfer ΔV_{imp}, semilatus rectum p_{tr}, and eccentricity e_{tr} of the transfer orbit for specific p_1, p_2, e_1, and e_2 of the boundary orbits.[13]

Conclusions

The canonical equations of Mayer's variational problem can be integrated analytically for IT arcs. Corresponding solutions represent the motion along spiral trajectories. The IT arcs satisfy the necessary condition of optimality if the constant of the Hamiltonian is positive and the radial component of the primer vector is negative. These arcs can be used in the optimal fixed-time transfer problem between coaxial elliptical orbits. However, the results show that this transfer by IT arcs is not possible between all arbitrary orbits. The angular variables at switching points depend only on the eccentricities and independently on semilatus recta and longitudes of pericenters.

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Chaos Analysis on Librational Control of Gravity-Gradient Satellite in Elliptic Orbit

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I. Introduction

LIBRATIONAL motion of a gravity-gradient satellite[1] in an elliptic orbit is governed by a system of nonlinear equations that become nonautonomous when the effect of orbital eccentricity is taken into consideration. It is well known that chaotic phenomenon can be observed in such a nonlinear and nonautonomous dynamic system over some ranges of the initial conditions. With advancement of mathematical and computational tools, chaos has been analyzed as a nonlinear phenomenon by such techniques as Poincare maps, Lyapunov exponents, reconstruction of attractors, and bifurcation diagrams.[2–4] Karasopoulos and Richardson[5,6] have applied these techniques to analyze the nonlinear dynamics of the gravity-gradient satellite system. Nixon et al.[7] and Fujii and Ichiki[8] have also applied these techniques to analyze the nonlinear dynamics of the tethered satellite system.

Recently, a few control concepts for chaos have been developed to convert chaotic behavior into a periodic or constant motion. Pyragas has proposed a control method called time-delay feedback control (DFC) in Ref. 9. This method is robust to small parameter variations and has been successfully used in several applications. Various modifications have also been considered in Ref. 10 for stabilizing stronger chaotic systems. Analysis of the system stability controlled by DFC has been discussed in Refs. 11 and 12. The main contribution of the present paper is an attempt to apply the DFC scheme successfully to the nonlinear dynamics of gravity-gradient satellites. It is shown that the control scheme is extremely effective for the present chaotic system.

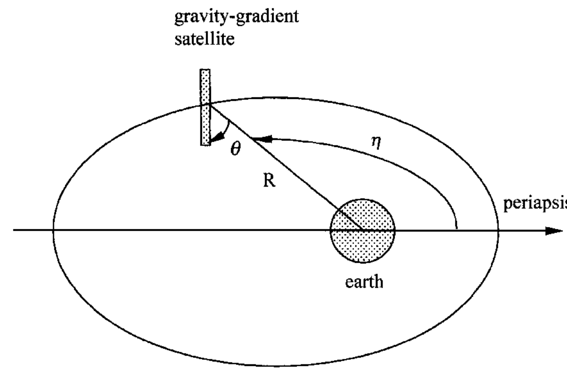

Fig. 1 System model

II. Dynamical Model

The dynamic system treated in this paper is a rigid gravity-gradient satellite in an elliptic orbit as shown in Fig. 1. The dynamic model is simplified by employing the following three assumptions:

1) Energy dissipation effects such as the aerodynamic drag will be ignored.

2) Only the pitch motion in the orbital plane is considered.

3) The inertia ratio of the satellite system is taken to be one, that is, the equal moments of inertia about the pitch and roll axes are significantly large compared to that about the yaw axis.

The equation of pitch motion for the rigid gravity-gradient satellite in an elliptic orbit can be described as follows:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2} = -\frac{3}{2(1+e\cos\eta)}\sin(2\theta) + \frac{2e\sin\eta}{1+e\cos\eta}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\eta}+1\right) - u(\eta) \tag{1}$$

where, θ, e, and $u(\eta)$ denote the pitch angle, the orbital eccentricity, and the control input, respectively, and the independent variable is the true anomaly η.

In general, chaotic motion means that the behavior of the system can be predicted for the short term but not all of the time, and the motion is regarded as chaotic if it simultaneously satisfies the following two characteristics. The first characteristic is sensitive dependence on the initial conditions. Chaos occurs in the deterministic system, but exhibits random behavior, that is, trajectories of a chaotic system starting from two nearly initial conditions will eventually separate and become uncorrelated, but always remain bounded in space. The second characteristic is topological transition. The trajectory can be close to any points in the bounded space, and such a trajectory is said to be dense. Consequently, chaos can be defined as trajectories that are neither equilibrium points nor periodic ones in a bounded space.

Concerning our dynamic system, the eccentricity can affect the nonlinear dynamics through the coefficients of nonlinear terms, and chaotic motion may occur in the situation when the eccentricity is taken into account, as in Eq. (1).

III. DFC

The two fundamental characteristics of chaos mentioned earlier are used for controlling the chaotic system. By the use of the characteristic of sensitive dependence on the initial conditions, it is possible to have large influence on the system dynamics by very small perturbations or external influence; moreover, the response of the system

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